Nonlinear Quadratic Pricing for Concavifiable Utilities in Network Rate Control

Quanyan Zhu
Coordinated Science Laboratory
Department of Electrical and Computer Engineering
University of Illinois at Urbana-Champaign
Email: zhu31@control.csl.uiuc.edu

Raouf Boutaba
School of Computer Science
University of Waterloo
Email: rboutaba@uwaterloo.ca

***Abstract*—This paper deals with a category of concavifiable functions that can be used to model inelastic traffic in the network. Such class of functions can be concavified within an interval of interest using a quadratic pricing term so that we obtain as a result a concave objective function. We use a game-theoretical framework as well as a centralized optimization approach to discuss the heterogeneous network with nonlinear quadratic pricing. We point out the equivalence between these two frameworks and use a Stackelberg player as an extra degree of freedom to design pricing policy for the network. In the end, we propose an auction-like iterative algorithm and illustrate it with a numerical example.**

I. Introduction

Recent interest in the area of congestion control and differentiation of services in Internet engendered a significant amount of literature on the issue of network rate control. Most of the research on this subject uses a utility-based approach where each user is assumed to have a utility function associated with him in terms of the rate allocated to the application. A well-studied type of utility functions is concave functions, which is well-suited to model elastic network applications such as file transfer and email. These applications have no real-time requirements and no rate constraints. Their utilities increase with the allocated rate, but the rate of increase decreases with respect to the rate. Even though the elastic model is sufficient for some network usage, with the increasing number of real-time applications, it becomes essential to explore the regime of a heterogenous network with the non-elastic applications. Such applications, for example, voice/video over IP, have certain data rate threshold, below which the application suffers a dramatic degradation of the quality of service (QoS). These services are less elastic than data services and a simple concave model doesn't suffice.

In [1], a sigmoidal-like function is used to model such inelastic behavior, giving rise to a non-concave analysis of the rate allocation problem. It appears that the centralized optimization problem becomes difficult to tackle without concavity of the objective function. On the other hand, in [2], a delayed concave function is used to preserve some degree of concavity in the utility function. However, the non-smoothness of the functions makes the analysis somewhat cumbersome, appearing less appealing for system-wide optimization.

We observe that the non-elastic behavior can be modeled by a class of concavifiable function. A sigmoidal-like function can actually be concavified by an extra quadratic term within an interval of interest. The concavification turns a non-concave problem into a more tractable concave one. The extra component also has its interpretation as a pricing term for utilizing the network. It is reasonable for the network to charge more when a non-elastic application requires higher equality of service. The study of pricing is commonly investigated in the game-theoretical framework as in [3]. However, a common assumption of those work is the elasticity of the applications and the linearity of the pricing term. A recent work by Shen and Basar in [4] extends [3] to the regime of nonlinear pricing. However, the analysis becomes difficult due to the technical difficulty in dealing with coupled constraints in games. Proposed recently in [5], a game Lagrangian approach is used to establish the equivalence between a potential game and a centralized optimization problem. With this, we are able to link the work of [3] and [4] to discuss the nonlinear pricing of concavifiable utility functions. In addition, the advances in the Stackelberg game with constraints also equipped us with a way to determine optimal pricing policies in the network.

The major contributions of this work are: (1) to extend the study of non-elastic functions to a broader class of concavifiable functions; (2) to use nonlinear pricing to study non-elastic applications; (3) to establish a connection between the game approach adopted in [4] and the optimization approach in [1], which also is related to a Nash Bargaining approach adopted in [2] and [6]. We focus on sigmoidal-like functions and develop an auction-like algorithm for the rate allocation using a coordinate descent method. Instead of asymptotic analysis performed in [4], we provide a concrete method to solve the optimal pricing problem and the rate allocation problem.

The paper is structured as follows. In section II, we introduce the class of concavifiable functions. In particular, we find a concavifier for the sigmoidal functions. In section III, we discuss two models: a game model and a centralized optimization model. We point out their relations and use Stackelberg framework to introduce a pricing design mechanism. Lastly, in section IV, we use coordinate descent method to develop an iterative algorithm and illustrate the process with a numerical example in section V. We conclude and point out some of the

978-1-4244-2324-8/08/$25.00 © 2008 IEEE.

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE "GLOBECOM" 2008 proceedings.

future work in VII.

II. Concavifiable Utility Functions

Network applications can be separated into two categories. One is elastic, for example, email, text file transfer, in which the satisfaction of users increases gradually with the assigned bandwidth. Utilities for such applications are usually modeled with a concave function. On the other hand, we have a class of non-elastic applications, for example, video streaming, multimedia applications, etc. These applications need to have a minimum bandwidth to operate satisfactorily and thus require a higher QoS.

In recent literature, most efforts are made to address elastic type of utilities [6]–[8]. For the case of non-elastic utilities, only a few attempts were made. In [2], a delayed quadratic function is used to model the non-elastic behavior. The utility function is concave within a certain interval, but the non-smoothness of the function is a challenge in that it requires an inconvenient piece-wise analysis in different rate intervals. In [1], the non-elastic behavior is approximated by a smooth Sigmoidal-like function. The non-concavity of the utility function imposes a challenge for network optimization. In [1], a well-known theory in non-convex programming is used. However, it gives rise to cumbersome analysis involving subgradients and non-convex analysis.

We observe that the Sigmoidal-like function actually falls into a class of concavifiable functions, recently characterized in [9]. A special feature of this class of functions is that it can be turned into a concave function if a concavifying term, usually in a quadratic form, is added. In network, this extra term can be viewed as a penalty to the non-concave utility function arising from a given network pricing scheme. The process of concavification potentially broadens the scope of utility functions that we can use for network convex optimization. It extends a possible investigation into different classes of utility functions that may represent different qualities of service within the network. In this paper, we focus on the sigmoidal-like functions, but the same theory can also apply to other concavifiable functions such as double sigmoidal-like functions (1) of a different kind of network application [10].

$$U_i = \mathrm{sgn}(x_i - h_i)\left(1 - \exp\left(-\left(\frac{x_i - h_i}{s_i}\right)^2\right)\right), \quad (1)$$

where $\mathrm{sgn}(\cdot)$ is the sign function, h_i is its center and s_i is the steepness factor.

The additional pricing term from the concavification shall not be seen as an overhead. It is actually necessary to include such terms so that the network can have a control or an incentive to regulate network behavior. Rather, the concavification kills two birds with one stone, dealing with the issue of concavity and the pricing at the same time. A commonly studied type of pricing is linear [3], [11]. Non-linear pricing is recently studied in [4] based on a game-theoretical framework. However, its investigation is too general for our purpose. Also, due to the assumption in [4] that the utility functions are

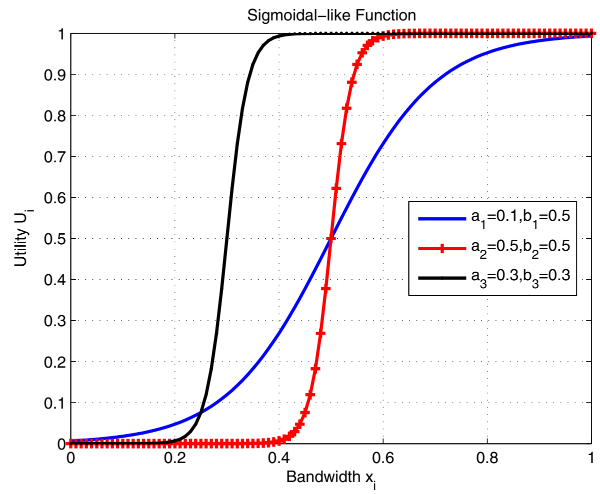

Fig. 1. Utility Functions

concave for elastic applications, its results are not directly applicable to rate control for non-elastic traffic.

We use the following two types of utility functions for two different traffic. $U_i^E(x_i) : \mathcal{R} \to \mathcal{R}$ and $U_i^N(x_i) : \mathcal{R} \to \mathcal{R}$ are given as follows.

$$U_i^E(x_i) = w_i \ln(1 + x_i), \forall i \in \mathcal{M}_E, \quad (2)$$

and

$$U_i^N(x_i) = \frac{1}{1 + e^{-a_i(x_i - b_i)}}, \forall i \in \mathcal{M}_N, \quad (3)$$

where $x_i \in A_i$, $A_i = [x_{\min,i}, x_{\max,i}]$, the set $\mathcal{M}_E$ denotes the set of elastic users; the set $\mathcal{M}_N$ denotes the set of non-elastic users, and $\mathcal{N} = \mathcal{M}_E \cup \mathcal{M}_N$ is the set of total users. It is easy to observe that both functions are increasing with respect to the allocated rate x_i. In particular, the utility function for the elastic users U_i^E is strictly concave but the one for nonelastic users is not. The function $U_i^N(x_i)$ is sigmoidal-like and has its inflection point at $x_i = b_i$; for $x_i > b_i$, the function is convex but for $x_i < b_i$, the function is concave. Figure 1 illustrates the shape the utility functions for different values of a_i and b_i. We can observe that a_i determines the curvature of U_i^N, but b_i translates the function along the $x-$axis, indicating the required rate level that needs to be satisfied for nonelastic applications.

A. *Concavification of Non-elastic Utility Function U_i^N*

Let $f : \mathcal{R}^N \to \mathcal{R}$ be a continuous function of N-variables on a compact set $A \subset \mathcal{R}^N$. The function f is said to be concave on A if

$$f(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \geq \lambda f(\mathbf{x}_1) + (1 - \lambda) f(\mathbf{x}_2), \quad (4)$$

for every $\mathbf{x}_1, \mathbf{x}_2 \in A$ and scalar $0 \leq \lambda \leq 1$. If f is twice differentiable, then f is concave if and only if $\nabla^2 f(\mathbf{x})$ is positive semi-definite at every $x \in \mathcal{R}^N$.

Definition 2.1: For a given continuous function $f : \mathcal{R}^N \to \mathcal{R}$ on a compact convex set A, consider the function $\phi(\mathbf{x}, \alpha)$:

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE "GLOBECOM" 2008 proceedings.

$\mathcal{R}^{N+1} \rightarrow \mathcal{R}$ defined by $\phi(\mathbf{x}, \alpha) = f(\mathbf{x}) - \frac{1}{2}\alpha \mathbf{x}^T \mathbf{x}$, where $\alpha \in \mathcal{R}$ is a scalar. If $\phi(\mathbf{x}, \alpha)$ is a concave function on A for some $\alpha = \alpha^*$, then $\phi(\mathbf{x}, \alpha)$ is said to be a concavification of f and α^* is its concavifier on A. A given function f is concavifiable if there exists a concave ϕ for some $\alpha^* \in \mathcal{R}$.

Assume that f is twice continuously differentiable, then we can characterize the concavifier explicitly. Denote $\mathbf{H}(\mathbf{x})$ as the Hessian matrix, i.e. , $H_{ij}(\mathbf{x}) = \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$. Also denote $\lambda(\mathbf{x})$ as the eigenvalue of matrix $\mathbf{H}(\mathbf{x})$. Since $\mathbf{H}(\mathbf{x})$ is symmetric, thus, $\lambda(\mathbf{x})$ is real. Let $\lambda^* = \max_{\mathbf{x} \in A} \lambda(\mathbf{x})$.

Theorem 2.1: (Zlobec, [9]) Given a twice continuously differentiable function $f : \mathcal{R}^N \rightarrow \mathcal{R}$ on a compact convex set $A \subset \mathcal{R}^N$. Then $\alpha = \lambda^*$ is a concavifier.

Theorem 2.2: (Zlobec, [9]) If α^* is a concavifier of f on a compact convex set A, then so is every $\alpha > \alpha^*$.

We call α a strict concavifier if $\alpha > \alpha^*$. Theorem 2.1 and Theorem 2.2 can be easily proven using the definition of concavity as in (2.1). Based on these fundamental results, a concavifier α_i^* can be found as follows as $U_i^N(x_i)$ is twice continuously differentiable.

$$\begin{aligned} \alpha_i^* &= \max_{x_i \in A_i} \frac{d^2 U_i^N(x_i)}{d^2 x_i} \\ &= \max_{x_i \in A_i} \frac{a_i^2 e^{a_i(b_i + x_i)} L_-(x_i)}{L_+^3(x_i)} \qquad (5) \\ &= \frac{19 + 11\sqrt{3}}{198 + 114\sqrt{3}} a_i^2 \\ &= 0.0962 a_i^2, \qquad (6) \end{aligned}$$

where $L_-(x_i) = e^{a_i b_i} - e^{a_i x_i}$ and $L_+(x_i) = e^{a_i b_i} + e^{a_i x_i}$. The optimal solution to (5) is obtained at $x_i^* = b_i - \frac{1}{a_i} \ln(2+\sqrt{3})$. It is achievable if $x_i^* \in A_i$, otherwise the maximum is achieved at boundary points $x_{\max,_i}$ and $x_{\min,_i}$. We assume $x_{\min,i}$ and $x_{\max,_i}$ is sufficiently small and large, i.e., $x_{\max,_i} > x_i^*$ and $x_{\min,_i} < x_i^*$. We can see that the optimal is only dependent on a_i, which governs the shape of the sigmoidal function. The larger a_i results in more abruptness in change and vice versa.

The concavifier can be thus obtained from (5). From Theorem 2.2, any $\alpha_i \geq \alpha_i^*$ is a concavifier. It is reasonable to interpret the concavifier as a pricing from (5). The user with a more desparate need for high bandwidth, i.e, large a_i, results in a higher pricing. We therefore, use this as a motivation to study quadratic pricing in the bandwidth allocation problem. We let the pricing term $C_i^N(x_i) = \frac{1}{2}\alpha_i(x_i - b_i)^2$ for user $i \in \mathcal{M}_N$ and $C_i^E(x_i) = \frac{1}{2}\alpha_i x_i^2$ for $i \in \mathcal{M}_E$. Figure 2 illustrates the sigmoidal-like functions concavified with the quadratic pricing terms.

III. Rate Control with Quadratic Pricing

In this section, we consider a rate allocation problem with capacity constraint $\sum_i x_i \leq C_0$. With multi-links, we may result in a constraint given by $\mathbf{A}\mathbf{x} \leq \mathbf{c}$, where $\mathbf{A}$ is a connection matrix that describes relation between users and links and $\mathbf{c} = [c_l]$ is a vector composed of capacity constraints on link l. We use a single-link to develop some insights

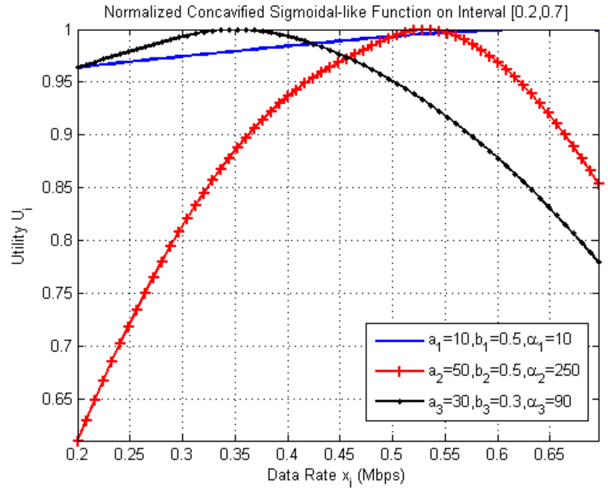

Fig. 2. Utility Functions Concavified with Quadratic Pricing

and simplify our analysis. The multi-link capacity problem is actually similar.

We denote the payoff function of each user as $P_i(x_i) = U_i^E(x_i) - C_i^E(x_i)$ for $i \in \mathcal{M}_E$ and similarly for inelastic users, we have $P_i(x_i) = U_i^N(x_i) - C_i^N(x_i)$, for $i \in \mathcal{M}_N$. Using nonlinear quadratic pricing term $C_i(x_i)$ naturally allows us to arrive at convex payoff functions for both classes of users. The process of concavification is also part of a price policy-making process that can be applied to any type of concavifiable utilities, including the Sigmoidal functions.

A. Game-model v.s. Centralized Optimization

We have two existing approaches of avail for the bandwidth allocation problem: one is based on centralized optimization and the other is using game-theoretical framework. The two approaches have been studied independently in [1] and in [4], respectively. In [1], the objective is to obtain an allocation for users that maximizes the total system utility, i.e.,

$$\text{(COP)} \qquad \max_{\mathbf{x}} \Phi(\mathbf{x}) = \textstyle\sum_{i \in \mathcal{N}} P_i(x_i)$$

$$\text{subject to} \qquad \textstyle\sum_{i \in \mathcal{N}} x_i \leq C_0, x_i \in A_i.$$

Since $P_i, \forall i \in \mathcal{N}$ is concave, the objective function in (COP) is also concave. Thus, (COP) can be solved using standard convex programming techniques.

In [4], a non-cooperative game $\Xi_{\mathbf{g}} = \langle N, A_i, P_i, \overline{\Omega} \rangle$ is considered, where $\overline{\Omega} = \{\mathbf{x} \in \mathcal{R}^N \mid \sum_i x_i \leq C.\}$. Since it is a non-cooperative game with coupled constraints, therefore we use Nash equilibrium to characterize the outcome of the game. It is defined as follows.

Definition 3.1: Let Ω denote $\Omega = \prod_{i \in \mathcal{N}} A_i$. A constrained NG $\Xi_{\mathbf{g}}$ with coupled inequality constraints $g_i(\mathbf{x}) \leq 0, i = \{1, \cdots, M\}$ or $\mathbf{g}(\mathbf{x}) \leq 0$ in vector form, with $\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}), \cdots, g_M(\mathbf{x})]^T$. A point $\mathbf{x}$ is feasible if $\mathbf{x}^* \in \overline{\Omega} \cap \Omega$. $\mathbf{x}^*$ is an NE solution to Ξ if

$$P_i(x_i^*, \mathbf{x}_{-i}^*) \geq P_i(x_i', \mathbf{x}_{-i}^*), \forall x_i' \in \overline{\Omega}_i(\mathbf{x}_{-i}^*), \forall i \in \mathcal{N},$$

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE "GLOBECOM" 2008 proceedings.

where $\overline{\Omega}_i(\mathbf{u}^*_{-i})$ is the projection set defined as

$$\overline{\Omega}_i(\mathbf{x}^*_{-i}) = \{x_i \in A_i \mid (x'_i, \mathbf{x}^*_{-i}) \in \overline{\Omega} \cap \Omega\}.$$

To solve for a Nash equilibrium in games with constraints may be quite involved. One of the approaches is to turn a constrained game into an unconstrained game by embedding a penalty term into the payoff function. A violation of the constraint results in a lower payoff. This approach is adopted in [4]. However, the analysis for a fixed point is quite involved. It is not practical in implementation, especially when we have multiple constraints. Instead, we adopt a game Lagrangian approach recently proposed in [5] and in [12]. Furthermore, using this approach, due to the separability, we can show that these two methods are actually equivalent.

Theorem 3.1: Solving (COP) for $\mathbf{x}^{\text{opt}}$ is equivalent to finding a NE $\mathbf{x}^*$ for $\Xi_{\mathbf{g}}$.

Proof: Since P_i is only a function of x_i, the objective function $\Phi(\mathbf{x})$ from (COP) is a potential function for the non-cooperative game, i.e.,

$$\frac{\partial P_i(x_i)}{\partial x_i} = \frac{\partial \Phi(\mathbf{x})}{\partial x_i}.$$

Since $\Phi(\mathbf{x})$ is concave, from Corollary 2.11 in [5], solving the constrained Nash potential game is equivalent to (COP), i.e., $\mathbf{x}^{\text{opt}} = \mathbf{x}^*$. Furthermore, if $\Phi(\mathbf{x})$ is strictly concave, we can guarantee the uniqueness of the solution. ■

Remark 3.1: From the perspective of efficiency, the price of anarchy ρ, defined in [13] as the ratio between game optimal and the social optimal, is always 100%, i.e.,

$$\rho = \frac{\Phi(\mathbf{x}^*)}{\Phi(\mathbf{x}^{\text{opt}})} = 1.$$

In other words, with the presence of coupled constraints, the game solution is as efficient as the centralized optimization.

B. Pricing Policy via Stackelberg Game

In the previous discussion, we assume that the pricing parameter is fixed for solving (COP) or the noncooperative game Ξ. In this section, we address the issue of pricing policy making via Stackelberg game. Using Stackelberg game for revenue-maximization has been discussed previously in [3]. We use this as a rationale to discuss the pricing policy making for the case with quadratic pricing.

Due to the equivalence between the game Ξ and (COP), we consider adding an extra freedom to the game framework by introducing a Stackelberg player, thus resulting in a $N+1$-person Stackelberg game. The Stackelberg player, denoted by S, can be interpreted as a higher-level network manager, whose rationality is to maximize his revenue by operating the network. From the pricing terms each user is subject to, we arrive at the following Stackelberg optimization problem (SOP) for the Stackelberg player.

$$\text{(SOP)} \quad \max_\alpha R(\mathbf{x}, \alpha) = \textstyle\sum_{i\in\mathcal{N}} C_i(x_i, \alpha_i)$$

$$\text{subject to} \quad \alpha \in \mathcal{R}^N_+. \quad \alpha_i \geq \alpha_i^*, i \in \mathcal{M}_N.$$

The constraints of the Stackelberg player come from the concavifier condition in Theorem 2.1. Due to the quadratic form of the cost terms, Stackelberg game can actually be solved using geometric programming as discussed in [14]. Another advantage of Stackelberg game is that it provides a way to deal with capacity constraints in the network. Since the solution of (SOP) depends on the pricing policy α, we can rely on the Stackelberg leader to design a pricing policy such that the constraints are not violated. By parameterizing the constraints with α, we can thus modify (SOP) by including the capacity constraints in modified Stackelberg optimation problem (MSOP).

$$\text{(MSOP)} \quad \max_\alpha R(\mathbf{x}, \alpha) = \textstyle\sum_{i\in\mathcal{N}} C_i(x_i, \alpha_i)$$

$$\text{subject to} \quad \alpha \in \mathcal{R}^N_+. \quad \alpha_i \geq \alpha_i^*, i \in \mathcal{M}_N. \quad \textstyle\sum_{i\in\mathcal{N}} x_i(\alpha_i) \leq C_0$$

Such approach simplifies the lower level game or optimization problem (COP) into an unconstrained problem. The Stackelberg player becomes a central manager who uses pricing as a control not only for revenue maximization but also for constraints satisfication.

IV. Parallel and Iterative Algorithm

We discussed in the previous section that the concavifer α_i^* can be used for two general purposes : (1) to penalize the demanding users via pricing; and (2) to concavify the sigmoidal-like utility function for network optimization. The first one provides a fair mechanism among all the users and the second nicely turns a non-convex optimization problem into a convex optimization problem.

Since the constrained game-theoretical framework coincides with the centralized optimization approach, we can use (COP) to derive a distributed and iterative algorithm. We use coordinate descent method, [15], instead of primal-dual approach due to two reasons as in [1] and [6]. One is the analytical difficulty of intractability of the sigmoid-like function when it comes to the formulation of the dual problem with nonlinear pricing; and the other is the auction-like feature of the coordinate descent algorithm makes it suitable for bandwidth allocation, as suggested in recent literature in [16], [17].

Let $\Phi(\mathbf{x}) = P_k(x_k) + P_{-k}$, where $P_{-k} = \sum_{i\neq k, i\in\mathcal{N}} P_i(x_i)$ is the sum of other user's utility. Since $\Phi(\mathbf{x})$ is separable in $x_i, i \in \mathcal{N}$. The gradient of Φ is given by $\nabla\Phi(\mathbf{x}) = \left[\frac{dP_i(x_i)}{dx_i}\right], i \in \mathcal{N}$. The gradient of the elastic users can be computed by

$$\nabla_i \Phi(\mathbf{x}) = \frac{w_i}{1+x_i} - \alpha_i x_i, \forall i \in \mathcal{M}_E, \tag{7}$$

and the gradient of the non-elastic users are given by

$$\nabla_i \Phi(\mathbf{x}) = \frac{a_i L_-(x_i)}{(1+L_-(x_i))^2}. \tag{8}$$

Using coordinate descent method, we update one component at a time. The update direction k is chosen such that

$$|\nabla\Phi(\mathbf{x})|_k = \|\nabla\Phi(\mathbf{x})\|_\infty. \tag{9}$$

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE "GLOBECOM" 2008 proceedings.

The steepest descent update for a given direction k is given by

$$\overline{x}_k = \text{argmax}\{P_k(x_k) \mid x_k \in A_k\}.$$

For the elastic user, $\overline{x}_k$ is found to be $\overline{x}_i = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4\xi_i}, \forall i \in \mathcal{M}_E$ where $\xi_i = \frac{w_i}{\alpha_i}$, if $x_{\min,i}$ and $x_{\max,i}$ are chosen such that $\overline{x}_i \in A_i$. The update $\overline{x}_i$ for elastic user is bounded from above such that $0 < \overline{x}_i \leq \sqrt{\xi_i}$. Therefore, we can ensure this by letting $x_{\min,i} = \epsilon$ and $x_{\max,i} > \sqrt{\xi_i}$. It can be noted that an increase in the pricing α_i results in a smaller optimal update $\overline{x}_k$. It is reasonable since, more expensive the network leads to a more conservative consumption of the bandwidth x_i.

On the other hand, for non-elastic users, the solution can be obtained by the first-order optimality condition. Since $P_i, i \in \mathcal{M}_N$ is concave with the quadratic pricing term, and thus, the first-order condition is both sufficient and necessary. Therefore we arrive at

$$\alpha_i(x_i - b_i) = \frac{a_i L_-}{(1+L_-)^2}, \forall i \in \mathcal{M}_N. \tag{10}$$

Finding $\overline{x}_k$ from (10) is more involved and demands solving a nonlinear equation. We use Newton-Raphson's method to search for $\overline{x}_i, i \in \mathcal{M}_N$. An iterative process is given by

$$x_i(n+1) = x_i(n) - \frac{P_i(x_i(n))}{P_i'(x_i(n))} \tag{11}$$

or more explicitly,

$$x_i(n+1) = x_i(n) - \frac{a_i g_i(n) L_+(x_i(n)) + \alpha_i L_+^3(x_i(n)) x_i(n)}{a_i^2 g_i(n) L_-(x_i(n)) + \alpha_i L_+^3(x_i(n))}, \tag{12}$$

or more explicitly, where $g_i(n) = a_i e^{a_i(b_i + x_i(n))}$. Since $P_i < 0$ and $P_i(x_i)$ is strictly concave on A_i if $\alpha_i > \alpha_i^*$, thus, $P_i(x_i)\frac{d^2 P_i(x_i)}{d^2 x_i} > 0, \forall x_i \in A_i$ and Newton-Raphson method yields a sequence $\{x_i(n)\}_{n=0}^{n=\infty}$ converging to $\overline{x}_i$.

We now can summarize the algorithm in the following.

Algorithm 4.1: **(Coordinate Descent Algorithm)**

Step 1: Initialize $\mathbf{x}$ by assigning $\mathbf{x}(0) = \mathbf{x}_0$, such that $\sum_{i\in\mathcal{N}} x_i \leq C$.

Step 2: Compute the gradient for $i \in \mathcal{N}$ according to (7) for $i \in \mathcal{M}_E$ and (8) for $i \in \mathcal{M}_N$.

Step 3: Select the largest component and its index k of the gradient vector $\nabla\Phi(\mathbf{x})$, such that (9) holds.

Step 4: Find $\overline{x}_k = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4\xi_k}$, if $k \in \mathcal{M}_E$, and otherwise, if $k \in \mathcal{M}_N$, we use the iterative algorithm in (11) to find $\overline{x}_k$.

Step 5: Update $\mathbf{x}(n+1)$ by

$$x_i(n+1) = \begin{cases} \min\{C - \sum_{j\neq i} x_i(n), \overline{x}_i(n)\}, & \text{if } i = k; \\ x_i(n), & \text{if } i \neq k. \end{cases} \tag{13}$$

Step 6: Iterate until $\|\mathbf{x}(n+1) - \mathbf{x}(n)\| \leq \delta$, where δ is a given tolerance of error.

The convergence of the coordinate descent algorithm has been discussed extensively in [15] and [18]. To ensure convergence, it usually requires to impose some rules on the order in which coordinates are iterated upon, for example, the almost-cyclic rule and the Gauss-Southwell rule.

The algorithm described in Algorithm 4.1 can be easily implemented in a parallel and distributed manner, where each user computes his gradient independently. In addition, we can view this algorithm as an auction, where the users are putting on their 'bids' with their calculated gradients, whereas the network behaves as an auctioner who picks the largest gradient and updates the allocation vector accordingly. Such process is repeated until an optimal solution is reached. The auction is open information in that the auctioneer broadcasts the result to the users at each iteration.

V. Numerical Example

In this section, we use a simple two-user single link example to illustrate the algorithm. Suppose we have two users who access one link with capacity constraint $x_1 + x_2 \leq 0.8$ Mbps. We let user 1 be an elastic user with its utility function given by $U_1 = \frac{1}{\ln 2}(1 + x_1)$ and user 2 be a non-elastic user with its utility U_2 given by a sigmoidal-like function with $a_2 = 50$ and $b_2 = 0.5$. The concavifier α_2 of U_2 needs to be chosen $\alpha_2 > \alpha_2^* = 240.5625$. We let the concavifier α_1 be the pricing for user 2. Under an unconstrained scenario, the allocation with respect to the pricing is illustrated in Figure 4. We observe that there is an almost linear relation between the allocated rate and the pricing for α_2 ranging from 250 to 500. The allocated resource decreases as the the price increases. However, the allocated rate is around 0.52-0.53 Mbps for the shown price range, which is beyond the mid-satisfaction level of user 2. Figure 3 illustrated the allocation with respect to pricing for user 1. For the price range from $\alpha_1 = 0.1$ to $\alpha_1 = 5$, the allocated rate drops more rapidly between $\alpha_1 = 0.1$ and $\alpha_1 = 1$ than from $\alpha_1 = 2$ to $\alpha_1 = 5$.

We choose $\alpha_1 = 2$ and $\alpha_2 = 400$, yielding $x_1 = 0.4856$ and $x_2 = 0.5229$, respectively from an unconstrained capacity. With the presence of the link capacity being 0.8, it is interesting that the constraint affects the allocation of the non-elastic user more than the elastic user. Using the Algorithm in section III, we obtain a fast convergence in two steps to $x_1 = 0.4856$ and $x_2 = 0.3144$ as illustrated in Figure 5. Such a convergence behavior is expected because we consider only two users and the feasible set is a simplex.

VI. Conclusion and Future Work

In this paper, we consider heterogenous users, elastic and inelastic, in network rate control. We include a nonlinear quadratic pricing term to concavify the sigmoidal-like function. Such approach results in a convex program (SOP) and its corresponding equivalent game Ξ, which can be solved using standard convex techniques. We also describe a Stackelberg framework for optimal pricing design in which the constraints from the concavification are imposed on the pricing parameters. With an extra degree of freedom, the Stackelberg framework can also be used for constraints regulation. In the end, we introduce an iterative algorithm that can be easily implemented in parallel like an auction.

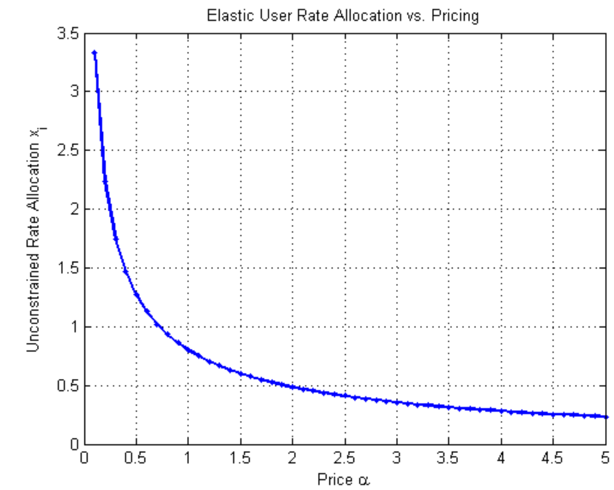

Fig. 3. Rate Allocation for User 1

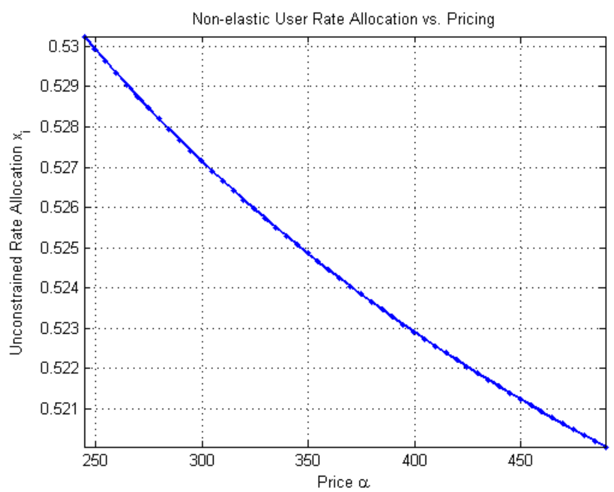

Fig. 4. Rate Allocation for User 2

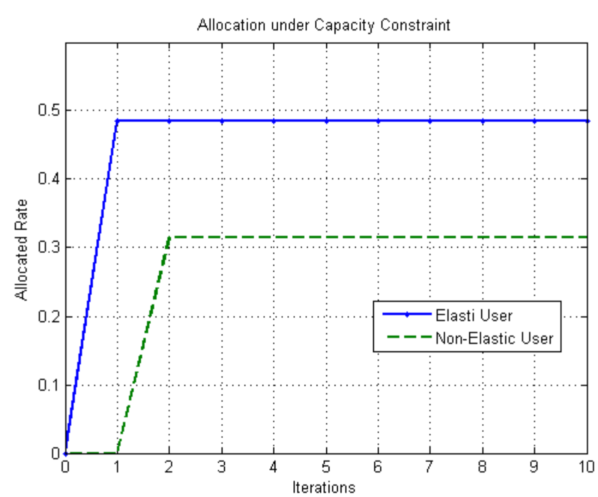

Fig. 5. Constrained Rate Allocation Subject to Link Capacity Constraint

In this paper, we introduce the notion of concavification and focus our discussion on the sigmoidal-like function proposed in [1]. In the light of current interest in fairness of allocation, we haven't discussed yet the property of proportional fairness of such type of utility function. In addition, we may also discuss other type of concavifiable functions, for example, arcsinh functions, which may appear to be more tractable due to the polynomial form of its derivative. It actually also can be shown that arcsinh function also meets the criterion of proportional fairness. Therefore, it might be worthy of investigation of such type of utility function in the future.

References

[1] J.-W. Lee, R. Mazumdar, and N. Shroff, "Non-convex optimization and rate control for multi-class services in the internet," *IEEE/ACM Transactions on Networking*, vol. 13, no. 4, pp. 827–840, August 2005.

[2] C.Touati, E. Altman, and J. Galtier, "Generalized Nash bargaining solution for bandwidth allocation," *Comput. Networks*, vol. 50, no. 17, pp. 3242–3263, 2006.

[3] T. Basar and R. Srikant, "Revenue-maximizing pricing and capacity expansion in a many-users regime," *Proc. IEEE Infocom 2002*, pp. 294 – 301, June 2002.

[4] H. Shen and T. Basar, "Optimal nonlinear pricing for a monopolistic network service provider with complete and incomplete information," *IEEE Journal on Selected Areas in Communications*, vol. 25, no. 6, pp. 1216–1223, August 2007.

[5] Q. Z. and, "A Lagrangian approach to constrained potential games, part I: Theory and examples," *Proceedings of Conference on Decision and Control*, 2008.

[6] H. Yaiche, R. Mazumdar, and C. Rosenberg, "A game theoretic framework for bandwidth allocation and pricing in broadband networks," *IEEE/ACM Transaction on Networking*, vol. 8, no. 5, pp. 667–678, 2000.

[7] S. Low and D. Lapsley, "Optimization flow control, I: Basic algorithm and convergence," *IEEE/ACM Transactions on Networking*, vol. 7, no. 6, pp. 861–874, 1999.

[8] F. Kelly, A. Maulloo, and D. Tan, "Rate control for communication networks: shadow prices, proportional fairness and stability," *J. Oper. Res. Soc.*, vol. 49, 1998.

[9] S. Zlobec, "Characterization of convexifiable functions," *Optimization*, vol. 55, no. 3, pp. 251–260, June 2006.

[10] T. Mitchell, *Machine Learning*. WCB-McGraw-Hill, 1997.

[11] L. Pavel, "A noncooperative game approach to OSNR optimization in optical networks," *IEEE Transactions on Automatic Control*, vol. 51, no. 5, pp. 848–852, May 2006.

[12] Q. Zhu and L. Pavel, "Constrained OSNR optimization in optical networks with a fictitious player," *IEEE Proc. of Broadnets*, 2007.

[13] T. Roughgarden and E. Tardos, "Bounding the inefficiency of equilibria in nonatomic congestion games," *Games and Economic Behavior*, vol. 47, pp. 389–403, 2004.

[14] Q. Zhu and L. Pavel, "End-to-end DWDM optical link power control via Stackelberg revenue-maximizing model," *Submitted to IEEE Control and Decision Conference (CDC)*, 2008.

[15] D. Bertsekas, *Network Optimization: Continuous and Discrete Models*. Athena Scientific, 1998.

[16] J. Huang, R. Berry, and M. Honig, "Auction mechanisms for distributed spectrum sharing," *42nd Allerton Conference on Communication, Control and Computing, Monticello*, September 2004.

[17] N. Semret, R. Liao, A. Campbell, and A. Lazar, "Peering and provisioning of differentiated internet services," *IEEE Infocom 2000*, March 2000.

[18] Z. Luo and P. Tseng, "On the convergence of the coordinate descent method for convex differentiable minimization," *Journal of Optimization Theory and Applications*, vol. 72, no. 1, pp. 7–35, 1992.